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Rossby wave packet interactions

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A mechanism is proposed whereby planetary zonal flows can be generated by the resonant interaction of Rossby wave packets whose amplitudes are slowly varying functions of both space and time. Equations are derived describing the long-time behaviour of a resonantly interacting triad. At the first closure certain properties analogous to those already known for discrete waves are deduced. At the second closure, in the particular case when one of the members of the triad is a zonal flow, it is shown that the sideband resonance mechanism can cause energy to be gained or lost by this zonal flow. It is also shown that a single Rossby wave packet can exchange energy with a zonal flow with weak shear. In the final section a resonant quarter mechanism for producing zonal flows is discussed. A numerical estimate of the acceleration of a zonal current from a zero initial state gives values of a few km/day per day.

1. Introduction

Non-linear wave interactions have been the subject of many studies in recent years. The first model used was one of discrete waves and Phillips (1960) showed that a gravity wave can be excited by three existing waves provided certain resonance conditions are satisfied. If the resonance conditions are not satisfied, then each wave undergoes only modal interactions which, in a conservative system, alter only the phase. In gravity waves an example of a non-linear modal interaction of a wave with itself is the well known Stokes wave. Interactions between a continuous spectrum of random gravity waves have been studied by Hasselmann (1962) who showed (using the Gaussian assumption) that the mechanism for energy interchange was the same as that proposed by Phillips but that the time scale required was longer. Subsequent analysis using the statistical approach by Benney & Saffman (1966), Benney & Newell (1969), showed that the Gaussian assumption was in fact unnecessary when dealing with any system supporting dispersive waves.

More recently, Benjamin & Feir (1967) while trying to produce the Stokes waves experimentally found that it was unstable. The cause of the instability is the excitation of sidebands of the basic mode at the expense of the discrete mode itself. Consequently, when dealing with wave interactions in systems which can support a continuous sectrum, the wave packet (group, train) concept is required. Mathematically, this is modelled by allowing the amplitudes of the waves to be slowly varying functions of position as well as time. Among the class of allowable functions are the discrete sidebands used by Benjamin & Feir. Using this approach it was shown by Benney & Newell (1967) that in most cases a discrete analysis is incapable of adequately predicting the long-time behaviour of these wave systems.

We now apply these ideas to a triad of Rossby wave packets. In this analysis we are not concerned with how the Rossby waves came to exist; but being a form of neutrally stable inertial oscillation on a rotating earth, we recognize that they can exist and that in the β-plane approximation their spectrum may be considered continuous. The waves probably arise from a baroclinic instability discussed by Charney (1959) and Miles (1964). A complete analysis should include a description of the generation of these waves from the instability of a shear flow and the resulting reinforcing of the shear flow by the action of the waves. However, our main concern here is with the latter problem, namely how energy in wave form can be converted into energy of the mean flow. It has been noted by Longuet-Higgins & Gill (1967) that if the waves are truly discrete, then due to the vanishing of the coupling coefficient, the triad resonance mechanism cannot be responsible for exciting zonal flows. The zonal flow merely acts as a catalyst for energy exchange between the other two members of the triad.

In contrast to the discrete wave approach, it will be shown that it is possible for the waves neighbouring the resonant waves to excite zonal flows on a longer time scale. A completely analogous situation is found to exist for a random continuous spectrum (Benney & Newell 1967) where, at the first closure, the feed to a zonal flow is zero due to the vanishing of the coupling coefficient. However, at the second closure a direct feed to the zonal flow is possible, arising from wave vectors in the local neighbourhoods of the resonance curves.

In the present analysis, the first closure gives the purely resonant interactions which do not feed zonal flows, whereas the second closure exhibits the sideband resonance. Numerical estimates, discussed in the appendix, of the acceleration of zonal currents yield accelerations of a few km/day per day. The possibility of planetary waves being the driving mechanism for the Cromwell current is also discussed.

Since the interaction with zonal flows occurs at times $t = O(1/\mu^2)$ (where μ is the order, of both the non-linearity and the packet spread), we must also consider quartet resonances where effect is first felt at this time scale. In the final section it is shown that it is possible for zonal flows to be excited in such a way.

2. Formulation of equations

We choose as our model the nondivergent β-plane approximation where (x, y) represent the horizontal co-ordinates measured in the east and north directions, respectively. Following Longuet-Higgins (1965) we write the components (east, north) of velocity (u, v)

$$u = \partial\psi/\partial y, \quad v = -\partial\psi/\partial x, \tag{2.1}$$

where

$$\psi = -g\xi/f \tag{2.2}$$

and ξ, g and f denote the surface elevation, gravitational acceleration and the Coriolis force ($2\Omega \cos\theta$; Ω being the earth's rotation and θ the co-latitude). A discussion of the validity of this approximation is given by Longuet-Higgins (1965) and we refer the reader to that paper for details.

From the conservation of potential vorticity following a fluid particle, we have

$$\frac{D}{Dt}\left(\frac{f-\nabla^2\psi}{h+\xi}\right) = 0, \tag{2.3}$$

where D/Dt denotes the substantial derivative and h the mean depth. This leads to the following differential equation for ψ, where only the non-linear terms representing the horizontal advection of vorticity have been retained consistent with the β-plane approximation:

$$\frac{D}{Dt}(\nabla^2-\alpha^2)\psi+\beta\frac{\partial\psi}{\partial x} = 0, \quad \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \tag{2.4}$$

where $$\alpha^2 = f^2/gh, \quad \beta = df/dy, \tag{2.5}$$

are treated as constants.

It is easily verified that (2.4) permits as an exact solution the westerly travelling wave (i.e. no second harmonics generated).

$$\psi \propto e^{i\theta}, \tag{2.6}$$

where $$\left.\begin{aligned}\theta &= k_x x+k_y y-\omega t, \\ \omega &= -\beta k_x/(\alpha^2+k^2).\end{aligned}\right\} \tag{2.7}$$

Longuet-Higgins & Gill (1967) examined the interaction of these discrete waves and noted that because of the existence of resonance conditions energy can be transferred from one component to another. The resonant conditions are that there exist classes of triads, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$, such that $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = 0$ and

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)+\omega(\mathbf{k}_3) = 0$$

are satisfied simultaneously. The existence of such triads in the case of Rossby waves was first pointed out by Kenyon (1964) and later with much detail by Longuet-Higgins & Gill (1967). However, it has been shown by several authors (Benjamin & Feir 1967; Whitham 1967; Phillips 1967; Benney & Newell 1967) that a discrete wave analysis may not be relevant due to the possible instability of neighbouring sidebands. It has also been noted by Benney & Newell (1967) that energy can be transferred to the side-bands of a discrete wave on the same time-scale as energy goes to the discrete wave itself by the non-linear resonant transfer mechanism.

Following the latter authors we begin by considering as our basic solution the three wave packets

$$\psi = \sum_{j=1}^{3} a_j(\mathbf{X}, T)\exp[i\theta_j]+(\text{complex conjugate}), \tag{2.8}$$

where the amplitudes a_j are slowly varying functions of position ($\mathbf{X} = \mu x$; $\mu \ll 1$) as well as time ($T = \mu t$). We restrict the a's to be at most bounded functions of $\mathbf{X}$ for large $\mathbf{X}$. The interesting balance occurs when the length-scale μ

corresponding to the package is of the same order of magnitude as the non-linear terms.

With the introduction of the additional time and length scales

$$T_1 = \mu t, \quad T_2 = \mu^2 t, \quad \mathbf{X} = \mu \mathbf{x}, \quad \mathbf{x} = (x, y); \tag{2.9}$$

the differential operators transform accordingly

$$\left.\begin{aligned} &\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial T_1} + \mu^2 \frac{\partial}{\partial T_2}, \\ &\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \mu \frac{\partial}{\partial x}, \\ &\nabla^2 \to \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2\mu \left(\frac{\partial^2}{\partial x \partial X} + \frac{\partial^2}{\partial y \partial Y}\right) + \mu^2 \left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right). \end{aligned}\right\} \tag{2.10}$$

Since these operators are to be applied to the functions $a(X, Y, T_1, T_2)\, e^{i\theta}$, we have

$$\begin{aligned} \frac{\partial}{\partial t} a\, e^{i\theta} &= \left(-i\omega + \mu \frac{\partial}{\partial T_1} + \mu^2 \frac{\partial}{\partial T_2}\right) a\, e^{i\theta} \\ &= \Omega a\, e^{i\theta}, \end{aligned}$$

where

$$\Omega = \omega + i\mu \frac{\partial}{\partial T_1} + i\mu^2 \frac{\partial}{\partial T_2}.$$

Similarly

$$\begin{aligned} \frac{\partial}{\partial x} a\, e^{i\theta} &= \left(ik_x + \mu \frac{\partial}{\partial x}\right) a\, e^{i\theta} \\ &= il_x a\, e^{i\theta}, \end{aligned}$$

where

$$l_x = k_x - i\mu(\partial/\partial x).$$

3. Analysis

The basic equation (2.4) reads

$$\frac{\partial}{\partial t}(\nabla^2 - \alpha^2)\psi + \beta \frac{\partial \psi}{\partial x} = \mu \left(\frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \nabla^2 \psi - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \nabla^2 \psi\right), \tag{3.1}$$

where the operators $\partial/\partial t$, $\partial/\partial x$, $\partial/\partial y$ are given by the transformation (2.10). It is easy to see that, with the basic solution consisting of the three packets with resonant phases $\theta_j = k_{jx}x + k_{jy}y - \omega_j t$, $\omega_j = \omega(\mathbf{k}_j)$ where $\theta_1 + \theta_2 + \theta_3 = 0$, the non-linear terms generate modes with the following phases:

$$\left.\begin{aligned} &\theta_2 + \theta_3 = -\theta_1, \quad \theta_3 + \theta_1 = -\theta_2, \quad \theta_1 + \theta_2 = -\theta_3, \\ &\theta_1 - \theta_2 = \theta_4, \quad \theta_3 - \theta_1 = \theta_5, \quad \theta_2 - \theta_3 = \theta_6. \end{aligned}\right\} \tag{3.2}$$

Thus we have three resonances and three new modes. We note that

$$\theta_4 = k_{4x}x + k_{4y}y - \omega^{(4)}t, \quad \mathbf{k}_4 = \mathbf{k}_1 - \mathbf{k}_2, \quad \omega^{(4)} = \omega_1 - \omega_2$$

and $\omega(k_4) = \omega_4 \neq \omega^{(4)}$. We use the superscript on the frequencies of the modes 4, 5 and 6 in order to emphasize that the frequency response of these modes is a forced and not a natural one.

We expand the dependent variable ψ

$$\psi = \psi_0 + \mu\psi_1 + \mu^2\psi_2 + \dots. \tag{3.3}$$

We begin with

$$\psi_0 = \sum_{j=1}^{3} a_j(\mathbf{X}, T) \exp[i\theta_j] + (\text{complex conjugate}), \tag{3.4}$$

which generates a ψ_1 containing three new modes

$$\psi_1 = \sum_{j=4}^{6} a_j \exp[i\theta_j] + (*), \tag{3.5}$$

$$\left.\begin{aligned} a_4 &= \frac{-i(\mathbf{k}_1 \times \mathbf{k}_2)(k_1^2 - k_2^2) a_1 a_2^*}{\omega^{(4)}(k_4^2 + \alpha^2) + \beta k_{4x}}, \\ a_5 &= -\frac{i(\mathbf{k}_3 \times \mathbf{k}_1)(k_3^2 - k_1^2) a_3 a_1^*}{\omega^{(5)}(k_5^2 + \alpha^2) + \beta k_{5x}}, \\ a_6 &= \frac{-i(\mathbf{k}_2 \times \mathbf{k}_3)(k_2^2 - k_3^2) a_2 a_3^*}{\omega^{(6)}(k_6^2 + \alpha^2) + \beta k_{6x}}, \end{aligned}\right\} \tag{3.6}$$

with the following side conditions chosen in order to remove secular terms:

$$\frac{\partial a_l}{\partial T_1} + \mathbf{c}_l \nabla a_l = \frac{(\mathbf{k}_m \times \mathbf{k}_n)(k_m^2 - k_n^2)}{\alpha^2 + k_l^2} a_m^* a_n^*, \tag{3.7}$$

where (l, m, n) is cycled over $(1, 2, 3)$; $\mathbf{c}_l$ is the group velocity of the wave

$$\mathbf{k}_l(k_{lx}, k_{ly}),$$

and $\mathbf{k}_1 \times \mathbf{k}_2$ is the vector product $(k_{1x}k_{2y} - k_{1y}k_{2x})$ and the operator ∇ is $(\partial/\partial x, \partial/\partial y)$. The asterisk (*) has been used to denote the complex conjugate.

If we had been dealing with discrete waves, the group velocity terms would be absent. Benney & Newell (1967) showed by looking at the stability of the exact solution

$$a_1 = a_1^{(0)}, \quad a_2 = 0, \quad a_3 = 0,$$

that there exists a wave number neighbourhood in which the side-bands of waves $\mathbf{k}_2$ and $\mathbf{k}_3$ can grow exponentially. Thus in dealing with wave packages where the package spread is the same order of magnitude as the non-linearity, (3.7) is the relevant equation.

Although no general solution of this equation set is known, we observe some properties which are somewhat similar to the case of discrete waves. We begin by noting that the operator $(\partial/\partial T_1) + \mathbf{a}.\nabla$ ($\mathbf{a}$ constant) acting on the function $g(\mathbf{X}, T)$ describes the change in $g(\mathbf{X}, T)$ along the straight line $\mathbf{X} - \mathbf{a}T_1 = \text{constant}$. Thus the left-hand sides of (3.7) denote the change of amplitude along group velocity lines $\mathbf{X} - \mathbf{c}_l T_1 = \text{constant}$, $l = 1, 2, 3$. If there were no non-linear interactions terms, then (3.7) would simply say that the amplitudes (equivalently energies) of the packages moved with their respective group velocities. We will denote $(\partial/\partial T_1) + \mathbf{c}_l.\nabla$ by $\mathscr{L}_l$. It is easily seen that

$$\mathscr{L}_1(\alpha^2 + k_1^2)^n a_1 a_1^* + \mathscr{L}_2(\alpha^2 + k_2^2)^n a_2 a_2^* + \mathscr{L}_3(\alpha^2 + k_3^2)^n a_3 a_3^* = 0, \quad (n = 1, 2). \tag{3.8}$$

For $n = 1$, the above relation says that the sum of the changes of energies of the three modes along their respective group velocity lines is zero; in the ease of discrete waves this relation can be integrated to yield that the total energy is constant in time. For $n = 2$, the above relation describes the 'conservation' of the square of the vorticity perturbation. We can also show

$$\mathscr{L}_1 \frac{(\alpha^2+k_1^2)\,a_1 a_1^*}{k_2^2-k_3^2} = \mathscr{L}_2 \frac{(\alpha^2+k_2^2)\,a_2 a_2^*}{k_3^2-k_1^2} = \mathscr{L}_3 \frac{(\alpha^2+k_3^2)\,a_3 a_3^*}{k_1^2-k_2^2}$$
$$= 2d\,\mathrm{Re}(a_l a_m a_n), \tag{3.9}$$

where $d = \mathbf{k}_1 \times \mathbf{k}_2 = \mathbf{k}_3 \times \mathbf{k}_1 = \mathbf{k}_2 \times \mathbf{k}_3$, since $\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3 = 0$.

Equations (3.9) are equivalent to the energy sharing equations of Benney (1962) developed for the case of four interacting discrete gravity waves. The equations relate the change in energy of one package along its group velocity lines to the changes of the other wave packages along their group velocity lines.

It was noted by Longuet-Higgins & Gill (1967) that the resonance mechanism of two discrete waves producing a third could not be responsible for producing time-independent zonal flows. One Fourier component of a zonal flow could be thought of as a wave $\mathbf{k}_3 = (0, -2k_y)$, as the natural frequency response of this is zero from (2.7). The waves that resonantly interact with this are $\mathbf{k}_1 = (k_x, k_y)$ and $\mathbf{k}_2 = (-k_x, k_y)$. However, since the moduli of these two waves are equal, we have

$$\left.\begin{aligned}
&\frac{\partial a_3}{\partial T_1}+\mathbf{c}_3\,.\,\nabla a_3 = 0,\\
&\frac{\partial a_1}{\partial T_1}+\mathbf{c}_1\,.\,\nabla a_1 = \theta a_2^* a_3^*,\\
&\frac{\partial a_2}{\partial T_1}+\mathbf{c}_2\,.\,\nabla a_2 = -\theta a_1^* a_3^*,
\end{aligned}\right\} \tag{3.10}$$

where
$$\theta = \frac{(\mathbf{k}_2\times\mathbf{k}_3)(k_2^2-k_3^2)}{\alpha^2+k_2^2},$$

and thus $\mathbf{k}_3$ acts as a catalyst for interaction between $\mathbf{k}_1$ and $\mathbf{k}_2$ but does not gain or lose energy itself at this time scale. We now propose to show that at a later time scale $t = O(1/\mu^2)$ through the action of resonating sidebands it is possible for zonal flows to be excited.

In order to see this we must proceed to the $O(\mu^2)$ balance in the governing equation (3.1) where in order to eliminate secular terms we must choose

$$\begin{aligned}
&\frac{\partial a_1}{\partial T_2}-\frac{i\omega_1^2}{\alpha^2+k_1^2}\nabla^2 a_1-\frac{2i}{\alpha^2+k_1^2}\mathbf{k}_1\,.\,\nabla\frac{\partial a_1}{\partial T_1}\\
&\quad=\frac{-i}{\alpha^2+k_1^2}[(k_2^2-k_3^2)\{a_3^*(\mathbf{k}_3\times\nabla)\,a_2^*-a_2^*(\mathbf{k}_2\times\nabla)\,a_3^*\}\\
&\qquad+2(\mathbf{k}_2\times\mathbf{k}_3)\{a_2^*\,k_3\,.\,\nabla a_3^*-a_3^*\,k_2\,.\,\nabla a_2^*\}]\\
&\qquad-\frac{-ia_1}{\alpha^2+k_1^2}\left[\frac{(\mathbf{k}_4\times\mathbf{k}_2)(k_4^2-k_2^2)(\mathbf{k}_1\times\mathbf{k}_2)(k_1^2-k_2^2)}{\omega^{(4)}(k_4^2+\alpha^2)+\beta k_{4x}}a_2a_2^*\right.\\
&\qquad\left.+\frac{(\mathbf{k}_5\times\mathbf{k}_3)(k_5^2-k_3^2)(\mathbf{k}_1\times\mathbf{k}_3)(k_1^2-k_3^2)}{\omega^{(5)}(k_5^2+\alpha^2)+\beta k_{5x}}a_3a_3^*\right],
\end{aligned} \tag{3.11}$$

plus two similar equations found by cycling (1, 2, 3) and (4, 5, 6). Use of the first closure (3.7) permits us to write

$$\begin{aligned}\frac{\partial a_1}{\partial T_2}-\frac{i}{2}Q^{(1)}_{XY}a_1 = \frac{-i}{\alpha^2+k_1^2}&[(k_2^2-k_3^2)\{a_3^*(\mathbf{k}_3\times\nabla)a_2^*-a_2^*(\mathbf{k}_2\times\nabla)a_3^*\}\\ &+2(\mathbf{k}_1\times\mathbf{k}_2)\{a_2^*k_3.\nabla a_3^*-a_3^*k_2.\nabla a_2^*\}]\\ &+\frac{2i}{\alpha^2+k_1^2}\left[\frac{(\mathbf{k}_2\times\mathbf{k}_3)(k_2^2-k_3^2)}{(\alpha^2+k_1^2)^2}\mathbf{k}_1.\nabla a_2^*a_3^*\right]\\ &-\frac{ia_1}{\alpha^2+k_1^2}\left[\frac{(\mathbf{k}_4\times\mathbf{k}_2)(k_4^2-k_2^2)(\mathbf{k}_1\times\mathbf{k}_2)(k_1^2-k_2^2)}{\omega^{(4)}(k_4^2+\alpha^2)+\beta k_{4x}}a_2a_2^*\right.\\ &\left.+\frac{(\mathbf{k}_5\times\mathbf{k}_3)(k_5^2-k_3^2)(\mathbf{k}_1\times\mathbf{k}_3)(k_1^2-k_3^2)}{\omega^{(5)}(k_5^2+\alpha^2)+\beta k_{5x}}a_3a_3^*\right], \quad \text{etc.,} \qquad (3.12)\end{aligned}$$

where $Q^{(1)}_{XY}$ is the dispersion tensor quadratic differential operator

$$\left[\frac{\partial}{\partial X},\frac{\partial}{\partial Y}\right]\begin{bmatrix}\dfrac{\partial^2\omega}{\partial k_x^2} & \dfrac{\partial^2\omega}{\partial k_y\partial k_y}\\ \dfrac{\partial^2\omega}{\partial k_x\partial k_y} & \dfrac{\partial^2\omega}{\partial k_y^2}\end{bmatrix}_{\mathbf{k}=\mathbf{k}_1}\begin{bmatrix}\dfrac{\partial}{\partial X}\\ \dfrac{\partial}{\partial Y}\end{bmatrix}. \qquad (3.13)$$

Noting that $$\frac{\partial}{\partial T_1}+\mu\frac{\partial}{\partial T_2}=\frac{\partial}{\partial T},$$

we have for the full closure describing the behaviour of the system for times $0\leqslant t\leqslant 1/\mu^2$,

$$\begin{aligned}\frac{\partial a_l}{\partial T}+\mathbf{c}_l.\nabla a_l-\frac{i\mu}{2}Q^{(l)}_{XY}a_l &= d_la_m^*a_n^*+i\mu f_l\{a_n^*(\mathbf{k}_n\times\nabla)a_m^*-a_m^*(\mathbf{k}_m\times\nabla)a_n^*\}\\ &\quad+i\mu g_l\{a_2^*\mathbf{k}_3.\nabla a_3^*-a_3^*\mathbf{k}_2.\nabla a_2^*\}\\ &\quad+i\mu h_l\mathbf{k}_l.\nabla a_m^*a_n^*+i\mu a_l(p_{lm}a_ma_m^*+p_{ln}a_na_n^*), \qquad (3.14)\end{aligned}$$

where (l,m,n) is cycled over (1, 2, 3) and $d_l, f_l, g_l, h_l, p_{lm}$ and p_{ln} may be read from (3.12).

The triad we wish to consider

$$\mathbf{k}_1=(+k_x,k_y),\quad \mathbf{k}_2=(-k_x,k_y),\quad \mathbf{k}_3=(0,-2k_y)$$

has the properties $k_1^2=k_2^2$, $\omega_3=0$, and thus

$$\mathbf{c}_3=(c_{3x},0),\quad \left.\frac{\partial^2\omega}{\partial k_x^2}\right|_{\mathbf{k}=\mathbf{k}_3}=\left.\frac{\partial^2\omega}{\partial k_x\partial k_y}\right|_{\mathbf{k}=\mathbf{k}_3}=\left.\frac{\partial^2\omega}{\partial k_y^2}\right|_{\mathbf{k}=\mathbf{k}_3}=0.$$

Also $$\mathbf{c}_1=(c_{1x},c_{1y}),\quad \mathbf{c}_2=(c_{1x},-c_{1y}),\quad \omega_2=-\omega_1,$$

$$d_2=-d_1,\quad f_3=0,\quad h_3=0,\quad p_{12}=p_{21}=0,\quad p_{32}=-p_{31}.$$

The equations (3.14) reduce to

$$\left.\begin{aligned}\left(\frac{\partial}{\partial T}+\mathbf{c}_1.\nabla\right)a_1-d_1a_2^*a_3^* &= \frac{i\mu}{2}Q^{(1)}_{XY}a_1+\mu\,\text{N.L. terms},\\ \left(\frac{\partial}{\partial T}+\mathbf{c}_2.\nabla\right)a_2+d_1a_1^*a_3^* &= \frac{i\mu}{2}Q^{(2)}_{XY}a_2+\mu\,\text{N.L. terms},\\ \left(\frac{\partial}{\partial T}+c_{3x}\frac{\partial}{\partial X}\right)a_3 &= i\mu g_3\{a_1^*\mathbf{k}_2.\nabla a_2^*-a_2^*\mathbf{k}_1.\nabla a_1^*\}.\end{aligned}\right\}\tag{3.15}$$

The relevant question to ask is whether the solutions to the set (3.15) differ only by $O(\mu)$ uniformly in time (for all times in the range $0 \leqslant T \leqslant 1/\mu$) from the solutions of the equations

$$\left.\begin{aligned}\left(\frac{\partial}{\partial T}+\mathbf{c}_1.\nabla\right)a_1^{(0)}-d_1a_2^{*(0)}a_3^{*(0)} &= 0,\\ \left(\frac{\partial}{\partial T}+\mathbf{c}_2.\nabla\right)a_2^{(0)}-d_1a_1^{*(0)}a_3^{*(0)} &= 0,\\ \left(\frac{\partial}{\partial T}+c_{3x}\frac{\partial}{\partial X}\right)a_3^{(0)} &= 0,\end{aligned}\right\}\tag{3.16}$$

or, equivalently posed, can we solve (3.15) by regular perturbation series in which the first terms are given by the solutions to (3.16)? Now ordinarily with second closures this is not the case, but we must ask the question for two reasons. The first is that the operator on the left-hand side tends to shift the waves off their basic resonance as the amplitudes move with the different group velocities whereas, secondly, the existence of non-trivial solutions to the first closure may preclude some terms of the second closure from being genuine secularities. By a genuine secular term we mean one which affects the order one term in the perturbation expansion.

First, let us consider the $a_l(\mathbf{X},T)\,(l=1,2,3)$ bounded functions for large $\mathbf{X}$ (e.g. periodic function, random function). Consequently we can view $a_l(\mathbf{X},T)$ as made up of a set of discrete $\{\exp[i\mathbf{k}_l\mathbf{X}]\}$ waves or as stationary random functions of position where we spectrally analyze the correlations $\langle a_l(\mathbf{R},T)\,a_l(\mathbf{R}+\mathbf{X},T)\rangle$. From either point of view we see the first response from (3.16) can be written

$$\exp[i(\mathbf{K}.\mathbf{X}-\nu T)],$$

where (treating $a_3^{(0)}$ as a constant)

$$\nu=\tfrac{1}{2}(\mathbf{c}_1+\mathbf{c}_2).\mathbf{K}\pm\tfrac{1}{2}[((\mathbf{c}_1-\mathbf{c}_2).\mathbf{K})^2+4d_1^2|a_3^{(0)}|^2]^{\frac{1}{2}},$$

or (if $a_3^{(0)}$ were zero initially)

$$\nu=\mathbf{c}_1.\mathbf{K}\quad\text{or}\quad\mathbf{c}_2.\mathbf{K}.$$

We could then generate resonance conditions, for if

$$a_1^{(0)}\propto\exp\{i(\mathbf{K}_1.\mathbf{X}-\mathbf{c}_1.\mathbf{K}_1T)\},$$

$$a_2^{(0)}\propto\exp\{i(\mathbf{K}_2.\mathbf{X}-\mathbf{c}_2.\mathbf{K}_2T)\},$$

$$a_3^{(0)}\propto\exp\{i(\mathbf{K}_3.\mathbf{X}-\mathbf{c}_3.\mathbf{K}_3T)\},$$

$a_3^{(1)}$ would grow as μT (random system would respond in time $T = O[1/\mu^2]$) if

$$\mathbf{K}_1+\mathbf{K}_2+\mathbf{K}_3 = 0, \quad \mathbf{c}_1.\mathbf{K}_1+\mathbf{c}_2.\mathbf{K}_2+\mathbf{c}_3.\mathbf{K}_3 = 0.$$

Essentially, one could view (3.15) as a coupled system supporting dispersive waves. However, it is well known that a *non*-dispersive system produces a much stronger non-linear response; this is perhaps best understood from the statistical viewpoint where the initial triple and higher correlations between waves do not decouple after long times if $\mathbf{k}_1+\ldots+\mathbf{k}_l = 0$ implies $\omega_1+\ldots+\omega_l = 0$. In fact, if the amplitudes $a(\mathbf{X},T)$ are decaying functions of $\mathbf{X}$ for large $\mathbf{X}$, thus permitting continuous Fourier integral transforms, then the only secular non-linear response arises when the waves are non-dispersive. We can make this choice by asking that the amplitudes are initially y-independent and by choosing the vector $\mathbf{k}_1$ such that $c_{3x} = c_{1x}$. This corresponds in the short wave case ($k^2 \gg \alpha^2$) to choosing the argument θ of $\mathbf{k}_1$ ($k_1\cos\theta$, $k_1\sin\theta$) to be 52·8°. For waves of arbitrary length the locus is given by the equation (real θ roots for $\lambda^2 < \sqrt{2}+1$)

$$8\sin^4\theta+(6\lambda^2-4)\sin^2\theta+(\lambda^4-2\lambda^2-1) = 0, \quad \lambda^2 = \alpha^2/k_1^2. \tag{3.17}$$

With the above choice and using the simple transformation

$$X' = X-c_{1x}T, \quad T' = T \tag{3.18}$$

the equation set (3.15) becomes

$$\left.\begin{aligned} \frac{\partial a_1}{\partial T'} &= d_1 a_2^* a_3^* + O(\mu), \\ \frac{\partial a_2}{\partial T'} &= -d_1 a_1^* a_3^* + O(\mu), \\ \frac{\partial a_3}{\partial T'} &= -i\mu k_x g_3 \frac{\partial}{\partial X'} a_1^* a_2^*. \end{aligned}\right\} \tag{3.19}$$

If we try to solve (3.19) by the regular perturbation series

$$a_l = a_l^{(0)}+\mu a_l^{(1)}+\ldots \quad (l = 1,2,3) \tag{3.20}$$

we obtain

$$\left.\begin{aligned} a_3^{(0)} &= a_3^{(0)}(X') \\ a_1^{(0)} &= \sum_{s=+,-} b^{(0)s}(X')\exp\{isd_1|a_3^{(0)}(X')|T'\}, \\ a_2^{(0)} &= \sum_{s=+,-} \frac{-is|a_3^{(0)}(X')|}{a_3^{(0)}(X')}\, b^{(0)*s}(X')\exp\{-isd_1|a_3^{(0)\prime}(X')\,T'\}. \end{aligned}\right\} \tag{3.21}$$

Clearly, in the product $(\partial/\partial X')a_1^{(0)*}a_2^{(0)*}$ there are terms which are independent of T' and thus on integration of

$$\partial a_3^{(0)}/\partial T' = f(X',T')+g(X')$$

we find that the perturbation series (3.20) becomes non-uniform in time. This merely indicates that the terms arising in the second closure do affect the perturbation series at the zeroth-order and must be considered as fully secular.

We summarize the results as follows: if the initial amplitudes are bounded but order one for large $\mathbf{X}$, then we expect that due to the possibility of secondary resonances we will get an energy transfer to the zonal flow on the time scale $t = O(1/\mu^2)$; if the initial amplitudes are decaying at large $\mathbf{X}$ such that they possess continuous Fourier integral transforms, then we expect an energy transfer to the zonal flow on the same time scale only when we make the choice (3.17). In an analysis of the interaction of random waves whose dispersion relation permitted triad resonances, Benney & Newell (1969) found that zonal flows could be excited at the second closure time scale $[(t = O(1/\mu^4)]$.

4. Breakdown of a single Rossby wave packet

Due to the fact that a single discrete wave is an exact solution of (3.1) we can treat the non-linear terms as the same order of magnitude as the linear ones when we look at the interaction of a wave packet and a zonal flow with weak shear,

$$\left.\begin{aligned}\psi_0 &= a(\mathbf{X},T)\,e^{i\theta}+a^*(\mathbf{X},T)\,e^{-i\theta}+b(\mathbf{X},T),\\ \psi &= k_x x+k_y y-\omega t, \quad b(\mathbf{X},T)\,\text{real}.\end{aligned}\right\} \tag{4.1}$$

The scales are defined as before

$$\mathbf{X} = \mu\mathbf{x}, \quad T_1 = \mu t, \quad T_2 = \mu^2 t, \tag{4.2}$$

where μ is a measure of the package spread.

The governing equation is the same as before except that ϵ, which measures the relative magnitude of the non-linear terms and which in the previous case was small, may now be order one.

$$\frac{\partial}{\partial t}(\nabla^2-\alpha^2)\,\psi+\beta\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi, \tag{4.3}$$

where $\partial/\partial t, \partial/\partial x, \partial/\partial y$ are given by the transformation (2.10). As before we make the perturbation

$$\psi = \psi_0+\mu\psi_1+\mu^2\psi_2+\dots. \tag{4.4}$$

We find at the $O(\mu)$ balance

$$\psi_1 = 0 \tag{4.5}$$

and in order to eliminate secular terms, we choose

$$\left.\begin{aligned}\frac{\partial a}{\partial T_1}+\mathbf{c}\,.\,\nabla a &= -\frac{ik^2}{\alpha^2+k^2}a(\mathbf{k}\times\nabla)\,b,\\ \frac{\partial b}{\partial T_1}+\mathbf{c}_0\,.\,\nabla b &= 0, \quad \mathbf{c}_0 = (-\beta/\alpha^2, 0).\end{aligned}\right\} \tag{4.6}$$

As before, at the first closure, the zonal flow is not affected but acts as a catalyst in changing the phase of $a(\mathbf{X}, T)$. Note that

$$\frac{\partial aa^*}{\partial T_1}+\mathbf{c}\,.\,\nabla aa^* = 0. \tag{4.7}$$

Eliminating secular terms at the order μ^2 balance yields the second closure

$$\left.\begin{aligned}\frac{\partial a}{\partial T_2}+\frac{i\omega}{\alpha^2+k^2}\nabla^2 a-\frac{2ik\nabla}{\alpha^2+k^2}\frac{\partial a}{\partial T_1} &= -\frac{2}{\alpha^2+k^2}[(\mathbf{k}\times\nabla)\,b]\,[\mathbf{k}.\nabla a]-\frac{k^2}{\alpha^2+k^2}\nabla a\times\nabla b,\\ \frac{\partial b}{\partial T_2} &= -\frac{2}{\alpha^2}(\mathbf{k}\times\nabla)\,\mathbf{k}\,.\,\nabla aa^*.\end{aligned}\right\} \tag{4.8}$$

Using (4.6) and the fact that

$$\partial/\partial T = \partial/\partial T_1 + \mu(\partial/\partial T_2),$$

we may write the full closure which will govern the behaviour of the system to times $t = O(1/\mu^2)$,

$$\left.\begin{aligned}\frac{\partial a}{\partial T} + \mathbf{c}.\nabla a - \frac{i\mu}{2} Q_{XY}\, a &= \frac{ik^2}{\alpha^2+k^2} a(\mathbf{k}\times\nabla) b \\ &\quad + \mu\left[-\frac{2}{\alpha^2+k^2}[(\mathbf{k}\times\nabla)b][\mathbf{k}.\nabla a]\right. \\ &\quad \left. + \frac{2k^2}{(\alpha^2+k^2)^2}\mathbf{k}.\nabla[a(\mathbf{k}\times\nabla)b] - \frac{k^2}{\alpha^2+k^2}\nabla a\times\nabla b\right], \\ \frac{\partial b}{\partial T} + \mathbf{c}_0.\nabla b &= -\frac{2\mu}{\alpha^2}(\mathbf{k}\times\nabla)\mathbf{k}.\nabla a a^*.\end{aligned}\right\} \tag{4.9}$$

Again, we give a simple example to show in general these equations cannot be satisfied by a regular perturbation series

$$a = a^{(0)} + \mu a^{(1)} + \ldots,$$
$$b = b^{(0)} + \mu b^{(1)} + \ldots,$$

where $a^{(0)}$, $b^{(0)}$ are solutions of the set

$$\left.\begin{aligned}\frac{\partial a^{(0)}}{\partial T} + \mathbf{c}.\nabla a^{(0)} &= -\frac{ik^2}{\alpha^2+k^2} a^{(0)}(\mathbf{k}\times\nabla) b, \\ \frac{\partial b^{(0)}}{\partial T} + \mathbf{c}_0.\nabla b^{(0)} &= 0.\end{aligned}\right\} \tag{4.10}$$

Equation (4.10) yields that

$$a^{(0)}a^{(0)*} = f(\mathbf{X} - \mathbf{c}T), \tag{4.11}$$

and suppose we choose as initial condition

$$a^{(0)}a^{(0)*}|_{T=0} = f\left(\frac{X}{c_{0x}-c_x} - \frac{Y}{c_{0y}-c_y}\right),$$

then

$$\begin{aligned} a^{(0)}a^{(0)*} &= f\left(\frac{X-c_xT}{c_{0x}-c_x} - \frac{Y-c_yT}{c_{0y}-c_y}\right) \\ &= f\left(\frac{X-c_{0x}T}{c_{0x}-c_x} - \frac{Y-c_{0y}T}{c_{0y}-c_y}\right) \\ &= g(\mathbf{X}-\mathbf{c}_0 T).\end{aligned}$$

Thus the simple transformation

$$\mathbf{X}' = \mathbf{X} - \mathbf{c}_0 T, \quad T' = T,$$

yields

$$\frac{\partial b^{(1)}}{\partial T'} = -\frac{2}{\alpha^2}(\mathbf{k}\times\nabla')\mathbf{k}.\nabla' g(\mathbf{X}')$$

and hence $b^{(1)}$ grows like T'.

We conclude that after a time $t = O(1/\mu^2)$ a fully non-linear wave package with a small spread can feed energy to a mean zonal flow with weak shear. However, since the physical quantities such as velocity depend on the gradients of the stream function, the order of magnitude of the velocities generated is less than that of the wave amplitude.

5. Quartet resonance mechanism

In this section we deal kinematically with the excitation of a zonal flow by a direct quartet resonance mechanism. Let us consider the four waves $\mathbf{k}_i(k_{ix}, k_{iy})$ $(i = 1, 2, 3)$ and $\mathbf{k}_4(0, k_{4y})$. From the existence of triad resonance, we know we can find triads $\{\mathbf{l}_i\}$ $(i = 1, 2, 3)$ such that

$$\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 = 0,$$

$$\omega(\mathbf{l}_1) + \omega(\mathbf{l}_2) + \omega(\mathbf{l}_3) = 0.$$

Let us now choose

$$\mathbf{k}_1 = \mathbf{l}_1 = (l_{1x}, l_{1y}),$$

$$\mathbf{k}_2 = \mathbf{l}_2 = (l_{2x}, l_{2y}),$$

$$\mathbf{k}_3 = (l_{3x}, -l_{3y}),$$

$$\mathbf{k}_4 = (0, 2l_{3y}).$$

It is clear that

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4 = 0.$$

Consider

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) + \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4)$$
$$= \omega(\mathbf{l}_1) + \omega(\mathbf{l}_2) + \omega(l_{3x}, -l_{3y}) + 0 = \omega(\mathbf{l}_1) + \omega(\mathbf{l}_2) + \omega(\mathbf{l}_3) = 0,$$

since $\omega(k_x, -k_y) = \omega(k_x, k_y)$ from (2.7). Thus at the order μ balance in (3.1) we generate non-secularily the following wave components

$$\exp\{i(\mathbf{k}_1 + \mathbf{k}_3) . \mathbf{x} - i(\omega_1 + \omega_3) t\}, \exp\{i(\mathbf{k}_2 + \mathbf{k}_3) . \mathbf{x} - i(\omega_2 + \omega_3) t\},$$

with coupling coefficients $(k_1^2 - k_3^2)(\mathbf{k}_1 \times \mathbf{k}_3)$ and $(k_2^2 - k_3^2)(\mathbf{k}_2 \times \mathbf{k}_3)$ respectively, and secularily the component

$$\exp\{i(\mathbf{k}_2 + \mathbf{k}_1) . \mathbf{x} - i(\omega_2 + \omega_1) t\}$$

as

$$\omega(\mathbf{k}_1 + \mathbf{k}_2) = \omega(\mathbf{l}_1 + \mathbf{l}_2) = -\omega(\mathbf{l}_3) = \omega(\mathbf{l}_1) + \omega(\mathbf{l}_2) = \omega(\mathbf{k}_1) + \omega(\mathbf{k}_2).$$

This non-uniformity is removed by the usual techniques described in §3.

However, at the $O(\mu^2)$ balance, the former two wave components react with the basic modes

$$\exp\{i\mathbf{k}_2 . \mathbf{x} - i\omega_2 t\}, \quad \exp\{i\mathbf{k}_1 . \mathbf{x} - i\omega_1 t\}$$

respectively, each coupling generating in secular fashion the Fourier component $\exp\{-ik_{4y} y\}$ of a zonal flow. The non-zero coupling coefficients are

$$\frac{-i(k_1^2 - k_3^2)(\mathbf{k}_1 \times \mathbf{k}_3)[(\mathbf{k}_1 + \mathbf{k}_3)^2 - k_2^2][(\mathbf{k}_1 + \mathbf{k}_3) \times \mathbf{k}_2]}{[\omega(\mathbf{k}_1) + \omega(\mathbf{k}_3)][\alpha_2 + (\mathbf{k}_1 + \mathbf{k}_3)^2] + \beta(k_{1x} + k_{3x})}$$

and

$$\frac{-i(k_2^2 - k_3^2)(\mathbf{k}_2 \times \mathbf{k}_3)[(\mathbf{k}_2 + \mathbf{k}_3)^2 - k_1^2][(\mathbf{k}_2 + \mathbf{k}_3) \times \mathbf{k}_1]}{[\omega(\mathbf{k}_2) + \omega(\mathbf{k}_3)][\alpha^2 + (\mathbf{k}_2 + \mathbf{k}_3)^2] + \beta(k_{2x} + k_{3x})}$$

respectively. We note that even though

$$\mathbf{l}_1 + \mathbf{l}_2 + \mathbf{l}_3 = 0$$

implies

$$(\mathbf{l}_1 + \mathbf{l}_3) \times \mathbf{l}_2 = 0,$$

then

$$(\mathbf{k}_1 + \mathbf{k}_3) \times \mathbf{k}_2 = (\mathbf{l}_1 + \mathbf{l}_3 - 2\mathbf{k}_4) \times \mathbf{k}_2 = -2\mathbf{k}_4 \times \mathbf{l}_2 \neq 0.$$

We conclude therefore that at the second closure we expect two types of terms forcing the zonal flows; the first is the sideband resonance mechanism dealt with in earlier sections and the second due to a quartet resonance.

6. Discussion

For a set of N interacting discrete waves in a weakly non-linear conservative physical system, the differential equations governing the amplitudes are well known. If the wave vectors are $\mathbf{k}_l$ ($l = 1, 2, \ldots, N$) with complex Fourier amplitudes $A(\mathbf{k}_l, t)$, these equations are:

$$\begin{aligned}\left(f(\mathbf{k}_l)\frac{d}{dt} + ig(\mathbf{k}_l)\right) A(\mathbf{k}_l, t) &= \epsilon \sum_{\mathbf{k}_m+\mathbf{k}_n+\mathbf{k}_l=0} K_1(\mathbf{k}_m, \mathbf{k}_n)\, A^*(\mathbf{k}_m, t)\, A^*(\mathbf{k}_n, t) \\ &+ i\epsilon^2 \left(\sum_p K_2(\mathbf{k}_l, \mathbf{k}_p - \mathbf{k}_p)\, A(\mathbf{k}_{l'} t)\, A(\mathbf{k}_{p'} t)\, A^*(\mathbf{k}_{p'} t)\right) \\ &+ \sum_{\mathbf{k}_q+\mathbf{k}_r+\mathbf{k}_s+\mathbf{k}_l=0} K_3(\mathbf{k}_{q'} \mathbf{k}_{r'} \mathbf{k}_s)\, A^*(\mathbf{k}_{q'} t)\, A^*(\mathbf{k}_{r'} t)\, A^*(\mathbf{k}_s, t) + O(\epsilon^3),\end{aligned} \qquad (6.1)$$

where $g(k_l)/f(k_l) = \omega_l$, the frequency of the linear system. Equation (6.1) is readily modified to describe wave packets when the complex amplitudes $A(\mathbf{k}_l)$ become slowly varying functions of space and time, namely

$$A_l = A(\mathbf{k}_l, t, \mathbf{X}, T_1, T_2),$$

where $\mathbf{X} = \mu\mathbf{x}$; $T_j = \mu^j t$ $(0 < \mu \ll 1)$. Using the multiple scale procedure, the original equation in physical space is modified by the transformations,

$$\frac{dt}{d} \to \partial/\partial t + \mu\, \partial/\partial T_1 + \mu^2\, \partial/\partial T_2, \quad \frac{\partial}{\partial x_\alpha} \to \frac{\partial}{\partial x_\alpha} + \mu \frac{\partial}{\partial X_\alpha}.$$

In Fourier space, the spatial transformation can be modified by setting

$$k_{l\alpha} \to k_{l\alpha} - i\mu \frac{\partial}{\partial X_\alpha}.$$

Thus taking the functions of $\mathbf{k}_l$ and expanding them in Taylor series

$$f(\mathbf{k}_l - i\mu\nabla) = f(\mathbf{k}_l) - i\mu \sum_{\alpha=1}^{2} \frac{\partial f}{\partial k_\alpha}\frac{\partial}{\partial X_\alpha} - \frac{\mu^2}{2} \sum_{\alpha\beta} \frac{\partial^2 f}{\partial k_{l\alpha}\partial k_{l\beta}} \frac{\partial^2}{\partial X_\alpha \partial X_\beta},$$

and, on performing the usual asymptotic analysis, we find the long-time behaviour of the spectral amplitudes are (for the case $\mu = \epsilon$, namely where the non-linearity balances the packet spread)

$$\begin{aligned}&\frac{\partial}{\partial T} a_l + \mathbf{c}_l \cdot \nabla a_l - \frac{i\mu}{2} Q^{(l)}_{xy} a_l + O(\mu^2) \\ &= \sum_{\substack{\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n=0 \\ \omega_l+\omega_m+\omega_n=0}} \alpha_{lmn} a_m^* a_n^* + \mu \Bigg[\sum_{\substack{\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n=0 \\ \omega_l+\omega_m+\omega_n=0}} \beta_{lmn} \{a_m^* \mathbf{f}_n \cdot \nabla a_n^* + a_n^* \mathbf{f}_m \cdot \nabla a_m^*\} \\ &\qquad + \sum_p \gamma_{lp} a_l a_p a_p^* + \sum_{\substack{\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n+\mathbf{k}_p=0 \\ \omega_l+\omega_m+\omega_n+\omega_p=0}} \delta_{mnp} a_m^* a_n^* a_p^* \Bigg] + O(\mu^2).\end{aligned} \qquad (6.2)$$

where $A_l = a_l e^{-i\omega_l t}$.

The left-hand side of (6·2) may be viewed as 'acceleration' terms describing the role of the group velocity vector and dispersion tensor; the right-hand side exhibits the non-linear forcing mechanism, the first due to direct triad resonances, the second to sideband triad resonances, the third to modal interactions and the fourth to direct quartet resonances. The second term is the one which has not appeared before in the literature and to whose effect the main part of this paper is directed. The truncated system ($\mu = 0$) will give the correct long-time behaviour of the system for times $T = O(1)$ (or $t = O(1/\mu)$), whereas for longer times $t = O(1/\mu^2)$, the full equation (6.2) must be solved. A solution with general initial values $a_l(\mathbf{X}, 0)$ is not known to this author. However, for the first closure in the case of interacting discrete waves, Bretherton (1964) has shown the solutions are periodic; Longuet-Higgins & Gill (1967) found elliptic function solutions for an interacting triad of Rossby waves under the special initial conditions which allow the sum of the phases of the complex amplitudes to be zero for all time. One may conjecture that at the first closure in an interacting triad of wave packets, one would find that the amplitudes were periodic functions along their group velocity lines. In a numerical scheme one can treat the group velocity lines as characteristics along which one can integrate from given initial values using the relations (3.8, 9).

If $\mathbf{k}_l$ represents a zonal flow $(0, k_{ly})$, then the coupling coefficient of the direct triad resonance mechanism is zero and thus the $O(\mu)$ terms in (6.2) are the important ones. In the appendix we examine the acceleration of a zonal flow from a zero state using the data of Eliasen (1958) and show that accelerations of a a few km/day per day are possible.

The possibility that the sideband resonances in internal waves are important is under investigation; in the Davis & Acrivos (1967) two-dimensional model, the coupling coefficient of the triad resonance producing a vertical shear flow is zero and thus the sideband mechanism may very well be important.

Appendix

We will discuss the case of the initial growth of a zonal flow due to the resonances with two discrete sidebands in the wave packets $k_1(k_x, k_y)$ and $k_2(-k_x, k_y)$. We find, using the data of Eliasen (1958) (also used by Kenyon (1966)) that zonal currents of a few kilometres per day can be generated.

Consider equation set (3.15) linearized about a basic state

$$\left.\begin{aligned} a_1^{(0)} &= A_1 \exp[i\mathbf{K}_1.(\mathbf{x}-\mathbf{c}_1 T), \quad a_2^{(0)} = A_2 \exp[i\mathbf{K}_2.(\mathbf{x}-\mathbf{c}_2 T)],\\ a_3^{(0)} &= 0. \end{aligned}\right\} \tag{A 1}$$

The third equation in this set becomes

$$\left(\frac{\partial}{dT} + c_{3x}\frac{\partial}{\partial x}\right) a_3 = \frac{-4\mu k_x k_y}{\alpha^2 + 4k_y^2}\{\mathbf{k}_2.\mathbf{K}_2 - \mathbf{k}_1.\mathbf{K}_1\} A_1^* A_2^* \times \exp[-i(\mathbf{K}_1 + \mathbf{K}_2).\mathbf{x} + i(\mathbf{c}_1.\mathbf{K}_1 + \mathbf{c}_2.\mathbf{K}_2) T].$$

The response of the particular sideband.

$$\mathbf{K}_3 = -\mathbf{K}_1 - \mathbf{K}_2, \quad a_3 = b_3 \exp[i\mathbf{K}_3.\mathbf{x}]$$

of the zonal flow is

$$\frac{\partial b_3}{\partial T} - c_{3x}(K_{1x}+K_{2x}) = \frac{-4\mu k_x k_y}{\alpha^2+4k_y^2}\{\mathbf{k}_2 . \mathbf{K}_2 - \mathbf{k}_1 . \mathbf{K}_1\} \exp\left[i(\mathbf{c}_1 . \mathbf{K}_1 + \mathbf{c}_2 . \mathbf{K}_2)\,T\right]. \tag{A 2}$$

Given K_1, choose K_2 such that

$$\mathbf{c}_1 . \mathbf{K}_1 + \mathbf{c}_2 . \mathbf{K}_2 = c_{3x}(K_{1x}+K_{2x}), \tag{A 3}$$

which yields the locus in the (K_{2x}, K_{2y}) plane

$$K_{2y} = \frac{c_{1x}-c_{3x}}{c_{1y}}(K_{2x}+K_{1x}) + K_{1y}. \tag{A 4}$$

This corresponds to the secondary resonance condition discussed earlier.

The response of the zonal flow is

$$a_3 = \text{A.F.} \exp\left[-i(\mathbf{K}_1+\mathbf{K}_2) . \mathbf{x} + i(\mathbf{c}_1 . \mathbf{K}_1 + \mathbf{c}_2 . \mathbf{K}_2)\,T\right], \tag{A 5}$$

where the amplification factor

$$\text{A.F.} = \frac{-4\mu^2 k_x k_y}{\alpha^2+4k_y^2}\{\mathbf{k}_2 . \mathbf{K}_2 - \mathbf{k}_1 . \mathbf{K}_1\}\, t$$

grows linearly with time. The amplification of the zonal velocity

$$\left(u_{\text{zonal}} = \frac{\partial \psi}{\partial Y}\right)$$

$$|\text{A.F.}| = t\left|\frac{8\mu^2 K_{3x}(\alpha^2+k^2)\,k_y(4k_y^2-k^2)}{(\alpha^2+4k_y^2)} A_1^* A_2^*\right|,$$

where we have used (A 4) and the values of $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{c}_1$, $\mathbf{c}_2$. Set

$$\mathbf{k} = (k\cos\theta, k\sin\theta), \quad v^2 = \alpha^2/k^2,$$

and the velocity amplification is

$$\left|8\mu^2 t\, K_{3x}\alpha^2 \frac{(1+v^2)\sin^2\theta(4\sin^2\theta-1)}{v^2(v^2+4\sin^2\theta)^2} A_1^* A_2^*,\right| \tag{A 6}$$

which reaches maximum value on the contours

$$\sin^2\theta = \frac{v^2}{4(1+2v^2)}, \tag{A 7}$$

on which contours, $$\text{A.F.} = \frac{\mu^2 t}{2} K_{3x}\frac{\alpha^2}{v^4}|A_1^* A_2^*|.$$

We use the data of Eliasen (1958)—also used and modified for a continuous random spectrum by Kenyon (1966)—and choose

$$k = \frac{5}{\text{earth radius}},$$

corresponding to the wavelength of the maximum energy waves at the 500 m.b. level in the atmosphere, $47\frac{1}{2}°$ north latitude. Choosing $h = 10^6$ cm, the scale height of the atmosphere, g gravity $= 10^3\,\text{cm/sec}^2$ and $f_0 = 2\Omega\cos 47\frac{1}{2} = 10^{-4}$ sec^{-1},

$$\alpha^2 = 10^{-17}\,\text{cm}^{-2} = 10^{-7}\,\text{km}^{-2}.$$

Thus $$v^2 = \alpha^2/k^2 = O(10^{-1}).$$

From (A 7) we conclude that the maximum amplification rate comes from waves whose wave vectors lie close to the east-west axis. In fact

$$\sin\theta \simeq \tfrac{1}{2}v = O(1/6).$$

We use Kenyon's non-isotropic spectrum (figure shown below) as this has maximum wave intensities in the region $\sin\theta = O(1/6)$.

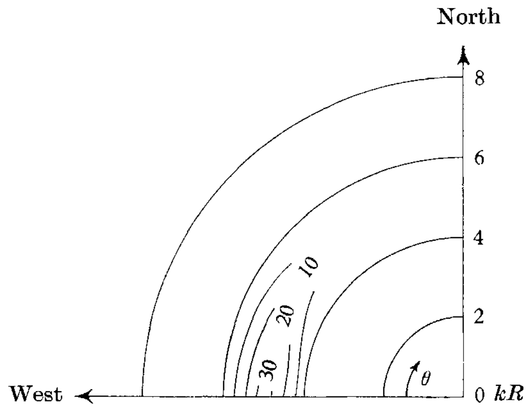

FIGURE 1. Initial energy spectrum $F(k)$; $\cos^4\theta$ spreading factor; contours of 10^{-7} F (km^3/day^2).

Assuming the initial amplitudes of wave modes 1, 2 to be the same,

$$\text{A.F.} = \frac{\mu^2 t}{2} O(k) \frac{\alpha^2}{v^4} |A|^2.$$

$\left(\text{Remember } K = O(k) \text{ as } \frac{\mu K}{k} = 0(\mu).\right)$ Also

$$k^2|A|^2 = \int_{\substack{\text{packet}\\ \text{spread}}} F(k)\,dk = F(k)\,\Delta k = F(k)\,\mu k,$$

where $F(k) = 3\times 10^8\,\text{km}^3/\text{day}^2$.

For the range of packet spread suggested by figure 1, $\mu = \frac{1}{6}$ to $\mu = \frac{1}{10}$ ($= O\{\sin\theta\}$), and for $v^2 = \frac{1}{10}$, we obtain an acceleration of the zonal current of 1–7 km/day per day. Thus, if the waves can exist for a number of days, they can produce appreciable zonal currents.

Let us make two further remarks to strengthen the hypothesis: first, as Kenyon points out, for these wave intensities, the non-linearity ϵ is larger than the package spread and would yield an extra factor ϵ/μ in the amplification rate; secondly, the effect of other possible modes (higher k, but same k_y) would be cumulative (could possibly also cancel) and even though the wave intensities for higher k (shorter waves) are smaller, the factor $1/v^4$ becomes larger for fixed α^2.

Note also, as we approach the equator $\alpha^2 \to 0$ and thus it would seem that the strongest zonal currents could be generated there provided there exist planetary-like waves of sufficient intensity in this region. We hesitate to put this forward as an explanation of the Cromwell current, (maximum velocity 100 cm/sec), but the possibility exists that long planetary-type ocean waves may be its driving mechanism. Perhaps the acid test would be to measure the slow spatial (east–west) and temporal behaviour of this current to see if it agrees with that suggested by (A 5).

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